Convex Hull

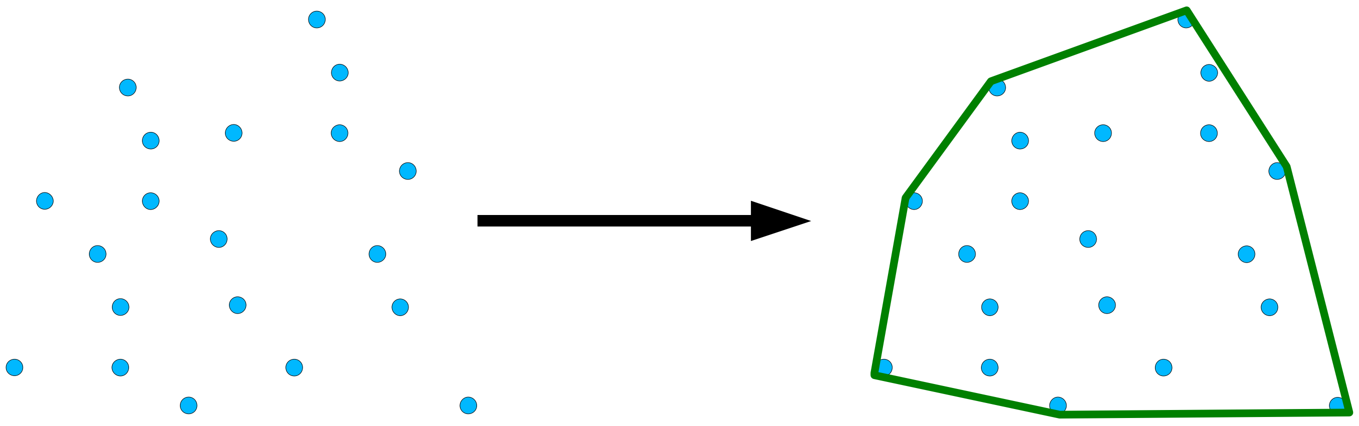
COMP 215 Lecture 5

Computational Geometry

- The area of CS concerned with solving geometric problems.
- Examples:
  - Finding intersections between line segments.
  - Finding closest pairs of points.
  - Finding the convex hull. (More on this in a second.)
- Uses in:
  - Graphics.
  - Robotics.
  - VLSI design.
  - etc.

Convex Hull

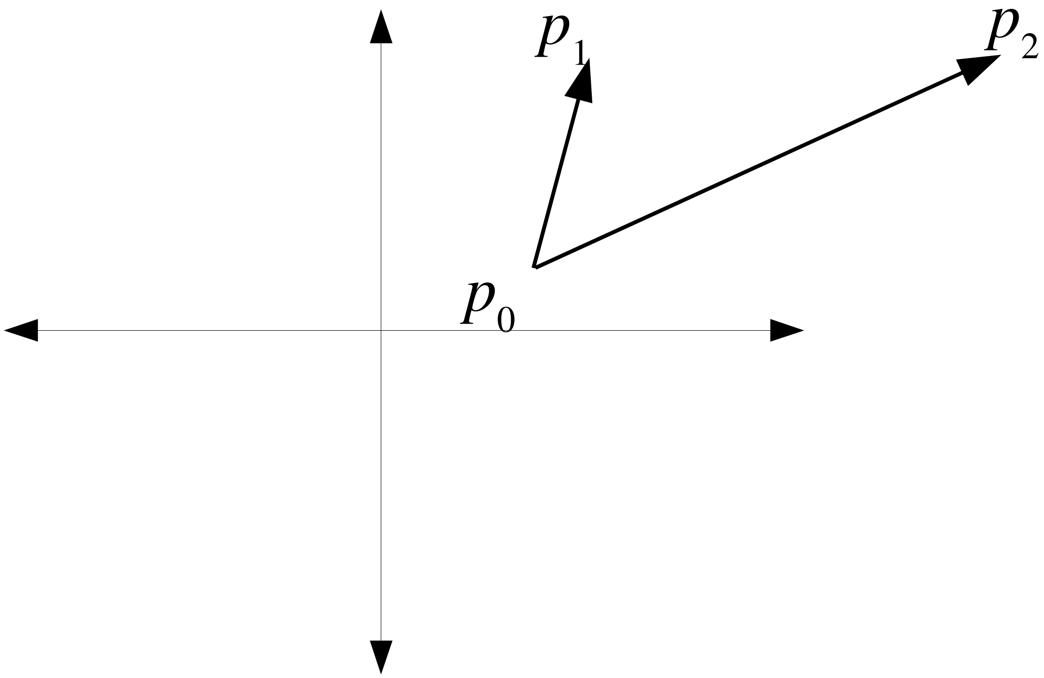
- The convex hull of a set Q of points is the smallest convex polygon P for which each point Q is either on the boundary of P or in its interior. (Introduction to Algorithms, Cormen et. al. 2001)



- The problem: For an arbitrary set of points Q, find the corresponding P.

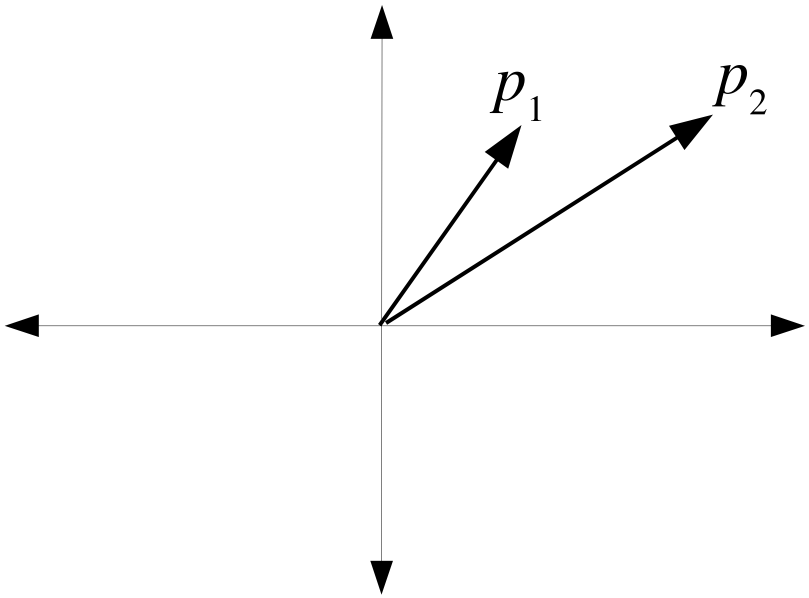
Line Segments Properties

- First question:
  - Given two directed line segments: $\overrightarrow{p_0 p_1}$ and $\overrightarrow{p_0 p_2}$, is $\overrightarrow{p_0 p_1}$ clockwise from $\overrightarrow{p_0 p_2}$?



Cross Product

- 2d cross product: $p_1 \times p_2 = x_1 y_2 - x_2 y_1$



- When this is positive p_1 is clockwise from p_2.
- When this is negative p_2 is clockwise from p_1.

Solution to Clockwise Problem

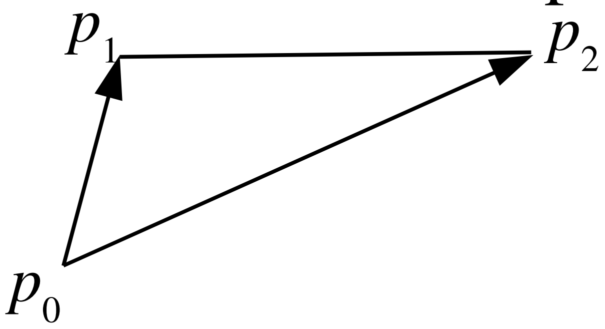
- The original question:
  - Given two directed line segments: $\overrightarrow{p_0 p_1}$ and $\overrightarrow{p_0 p_2}$, is $\overrightarrow{p_0 p_1}$ clockwise from $\overrightarrow{p_0 p_2}$?
- The solution: move p_1 and p_2 to use p_0 as the origin, and calculate cross product:

$$(p_1 - p_0) \times (p_2 - p_0) = (x_1 - x_0)(y_2 - y_0) - (x_2 - x_0)(y_1 - y_0)$$

- If this is positive then $\overrightarrow{p_0 p_1}$ is clockwise from $\overrightarrow{p_0 p_2}$.

Clockwise Turns

- Next question: do two consecutive line segments $\overline{p_0 p_1}$ and $\overline{p_1 p_2}$ make a clockwise, or counterclockwise turn at p_1?
- This is almost the same as the previous question:



$$(p_2 - p_0) \times (p_1 - p_0) = (x_2 - x_0)(y_1 - y_0) - (x_1 - x_0)(y_2 - y_0)$$

- Positive is a clockwise turn, negative is counterclockwise.

Back To Convex Hull

- Any ideas for a good algorithm?

Candidate Algorithm

- First, sort all points by their x coordinate.
  - (Theta(n lg n) time)
- Then divide and conquer:
  - Find the convex hull of the left half of points.
  - Find the convex hull of the right half of points.
  - Merge the two hulls into one. (this is the tricky step.)

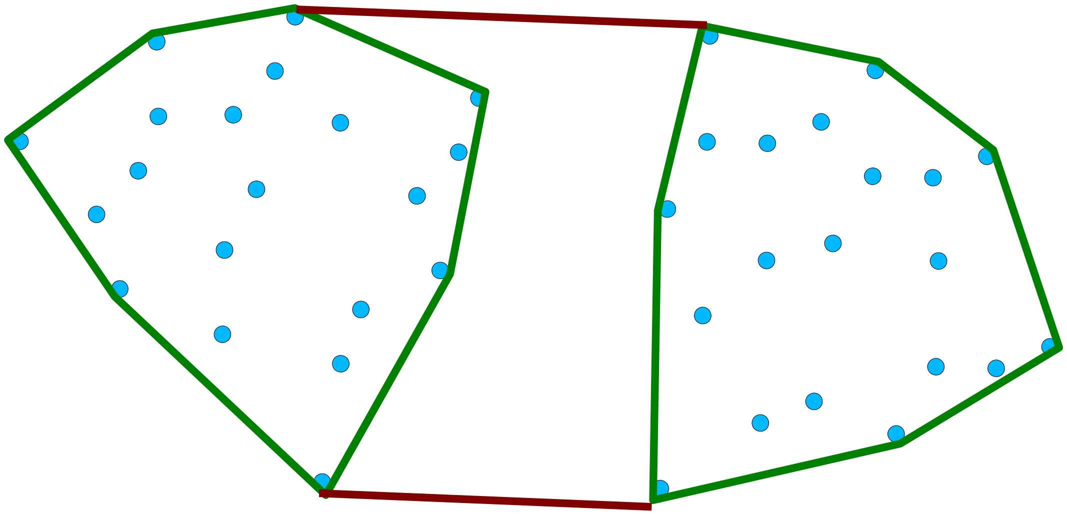
Convex Hull Pseudocode

```
//input: the number of points n, and
//an array of points S, sorted by x coord.
//output: the convex hull of the points in S.

point[] findHullDC(int n, point S[]) {
   if (n > 1) {
      int h = floor(n/2);
      m = n-h;
      point LH[], RH[]; //left and right hulls
      LH = findHullDC(h, S[1..h]);
      RH = findHullDC(m, S[h+1..n]);
      return mergeHulls(LH.size(), RH.size(),
                            LH, HR);
   } else {
      return S;
   }
}
```

Merging Hulls

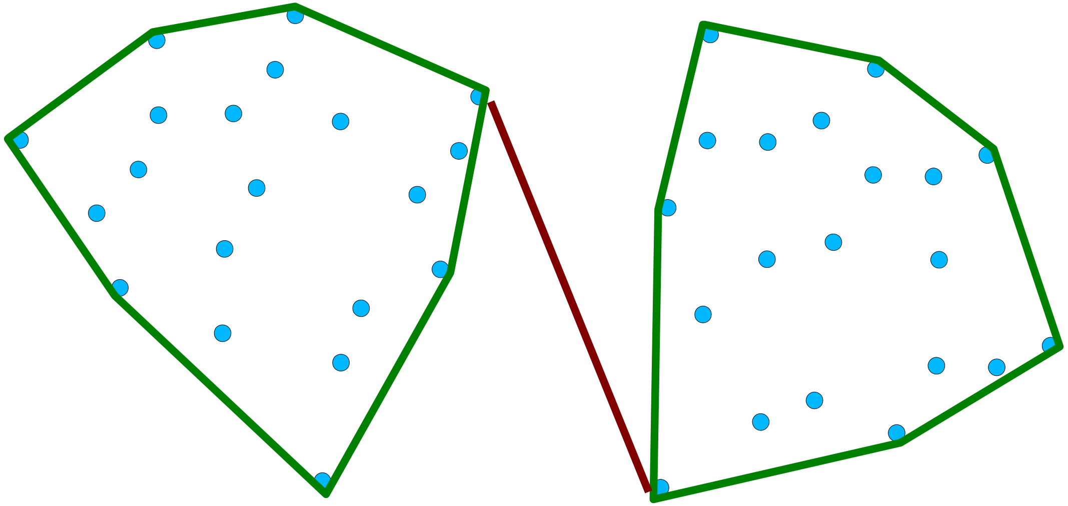
- Big picture:
  - first find the lines that are upper tangent, and lower tangent to the two hulls (the two red lines)



  - Then remove the points that are cut off.

Finding Tangent Lines

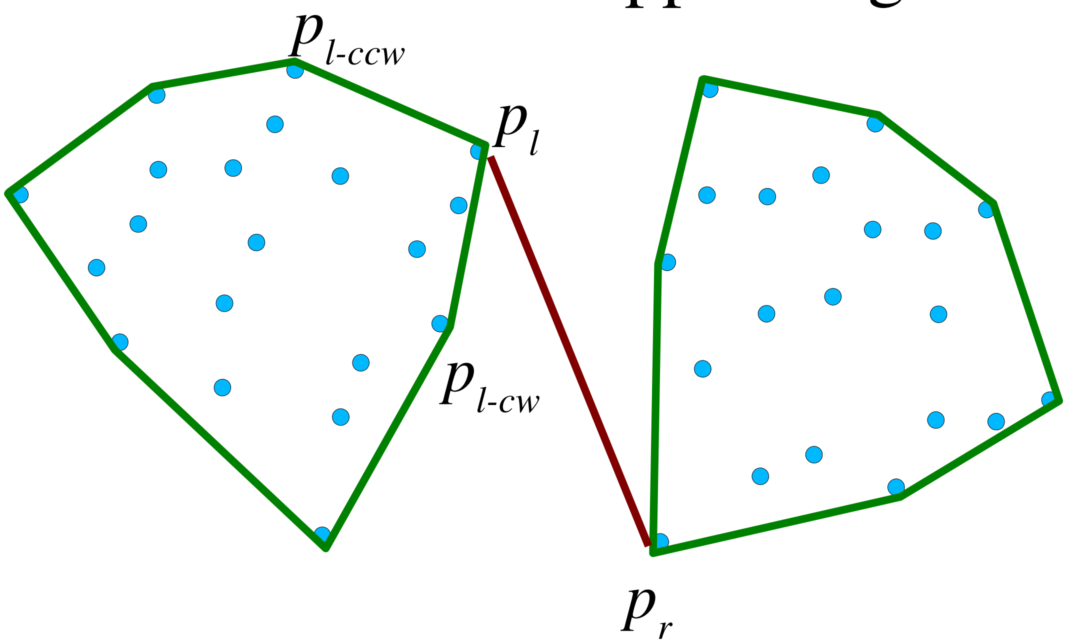
- Start with the rightmost point of the left hull, and the leftmost point of the right hull:



- While the line is not upper tangent to both left and right:
  - While the line is not upper tangent to the left, move to the next point (counter-clockwise).
  - While the line is not upper tangent to the right, move to the next point (clockwise).

Checking Tangentness

- How can we tell if a line is upper tangent to the left hull?



- The pair of line segments $\overline{p_r p_l}$, and $\overline{p_l p_{l\text{-}ccw}}$ should make a CCW turn at p_l.
- The same goes for $\overline{p_r p_l}$ and $\overline{p_l p_{l\text{-}ccw}}$.

The Tricky Bits

- Hulls need to be maintained in order (CW or CCW).
- Needs to be stored in a data structure that allows wrapped forward and backward iteration.
  - Circularly linked list.
  - Array with clever indexing.
- Several ways to handle base cases:
  - Special code to create hulls of size 1,2, and 3?
  - Clever merging that can merge a hull of size 2 with a hull of size 1? (or 1 and 1, or 3 and 2, etc.)

Analysis & PP

- Let's talk about running time.
- Then let's talk about the programming project.